Published by Institute of Physics Publishing for SISSA/ISAS

Received: *October 1, 2003*
Revised: *January 19, 2004*
Accepted: *February 1, 2004*

Chern-Simons matrix models and unoriented strings

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Abstract: For matrix models with measure on the Lie algebra of SO/Sp, the sub-leading free energy is given by $F_1(S) = \pm\frac{1}{4}\frac{\partial F_0(S)}{\partial S}$. Motivated by the fact that this relationship does not hold for Chern-Simons theory on S^3, we calculate the sub-leading free energy in the matrix model for this theory, which is a gaussian matrix model with Haar measure on the group SO/Sp. We derive a quantum loop equation for this matrix model and then find that F_1 is an integral of the leading order resolvent over the spectral curve. We explicitly calculate this integral for quadratic potential and find agreement with previous studies of SO/Sp Chern-Simons theory.

Keywords: Topological Field Theories, D-branes, Superstring Vacua.

Contents

1. Introduction

Matrix models have been studied intensely since the classic paper [1], where it was realized that they enumerate planar diagrams. Remarkably, it was recently discovered that there is a description of Chern-Simons (CS) theory [2, 3] and Holomorphic Chern-Simons (HCS) theory [4, 5, 6] on certain manifolds as particular matrix models. Whilst interesting in its own right, this discovery also has deep implications for supersymmetric gauge theory in four dimensions.

The connections between CS/HCS theory and field theory in four dimensions is the following (see [7] for a review). Type II string theory on a Calabi-Yau (CY) manifold can be twisted in one of two ways to give a topological string, the so called A and B models [8].

Furthermore, it was shown in [9] that the open string A-model on the CY manifold T^*M, where M is a lagrangian submanifold, is equivalent to CS theory on M with gauge group determined by the Chan-Paton factors. It was also shown that the open string B-model on a CY is equivalent to HCS theory on that CY. So calculating the partition function of these CS/HCS theories amounts to solving the topological A/B string.

For certain topological correlators the A/B twist is trivial and thus these topological correlators give physical correlators, this in fact happens for precisely the correlators that correspond to F-terms in the resulting four dimensional gauge theory [8]. So the conclusion is that the superpotential can be computed entirely from the topological string. This was made extremely precise in [10, 6], where it was shown that the effective superpotential in four dimensions W_{eff} is given by,

$$W_{\text{eff}} = N\frac{\partial F_0}{\partial S} - \tau S\,, \tag{1.1}$$

where, N is the amount of RR flux, F_0 is the leading order contribution to the topological string free energy, S is the gaugino condensate and τ is the gauge coupling. So showing that topological strings can be described by matrix models is very suggestive that the matrix model structure can be uncovered directly in field theory. For the case of $\mathrm{SU}(N)$, $\mathcal{N}=1$ gauge theory with adjoint matter, this was found in [11, 12].

After the initial work of Dijkgraaf and Vafa, it was shown how to generalize their work to the other classical gauge groups from several points of view. Matrix model generalizations were considered in [13, 14], the perturbative supergraph techniques of [11] were considered in [15] and generalized Konishi anomaly techniques of [12] were considered in [16, 17]. All these works were studying $\mathcal{N}=1$ SO / Sp gauge theories in four dimensions with adjoint matter and single trace superpotential.

The present work is concerned with how the calculations of [13] (which will be reviewed in section 2) can be performed in the CS model matrix model of [2, 3]. The main result of this paper is the calculation of F_1 in the CS theory on S^3 with gauge group SO / Sp. The partition function of this CS theory has in fact been calculated to all orders in [18], the present work explores by explicit calculation, the matrix model description of CS theory. At first glance there appears to be an inherent contradiction between a naive extrapolation of the results of [13] and the known partition function [18]. We will find in this paper that although the CS case is more complicated than the Lie algebra matrix model, the matrix model realizations of CS is not incorrect.

In [3], it was shown that the CS model matrix model is of the same type as the B-model matrix model but with a rather complicated double trace potential. It must be because of the double trace that the B-model calculation does not translate to the CS model.

This paper is organized as follows. In section 2 we review the calculation of [13] and present a new way to obtain the same results. This new method will generalize to the CS matrix model. In section 3 we describe the topological string and Calabi-Yau geometry which is inherently being studied. Section 4 will contain a discussion of the free energy of SO / Sp CS theory, what the naive contradiction is and what we will calculate from the matrix model. Section 5 will contain a derivation of the loop equation for matrix models

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with Haar measure, something which has not appeared in the literature. We will see explicitly why the method of [13] breaks down. In section 6 we will calculate the leading order free energy of the CS matrix model for groups SO / Sp, this is a straightforward generalization of [3] but is included for completeness. In section 7 we will calculate the subleading free energy, this is the main result of the paper. In section 8 we will discuss the four dimensional gauge theory which this analysis actually corresponds to.

2. Matrix models for classical groups

Matrix models for all the classical groups were first considered in [19], where they wrote down the appropriate measures in eigenvalue form. More recently, the authors of [13] studied the orientifolded CY geometry of [4] and the resulting matrix model. As is well known, orientifolding internal geometries in string theory leads to the gauge groups SO / Sp. It was shown in [13] that the relevant matrix model, when in eigenvalue form, has a measure on the Lie algebra of SO / Sp, as considered previously in [19].

2.1 A first look at F_1

Techniques for calculating $\mathcal{O}(N^{-2})$ and higher corrections to the hermitean matrix model free energy were considered previously in [20, 21](they correspond to four dimensional gravitational F-terms). The same method was used in [13] to derive the $\mathcal{O}(N^{-1})$ correction to the free energy (F_1) and will now be reviewed. First, lets set up some notation. The partition function of the matrix model is,

$$\mathcal{Z} = \int d\Phi \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}W(\Phi)}\,, \tag{2.1}$$

where, $W = \sum_{j=1}^{\infty} \frac{g_j}{2j}\Phi^{2j}$, and Φ is in the adjoint of SO / Sp. For a single cut model, the number of eigenvalues in that cut is M, and we define $S = g_s\frac{M}{2}$. The resolvent is defined as, (g is for genus, c is for crosscap)

$$\omega(x) = g_s\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle = \sum_{g,c} g_s^{2g+c}\omega_{2g+c}(x)\,. \tag{2.2}$$

Now we need the relation,

$$\omega(x) = \frac{d}{dV}(x)F + \frac{S}{x} \tag{2.3}$$

where $\frac{d}{dV}(x) = -\sum_{j=1}^{\infty}\frac{2j}{x^{2j+1}}\frac{\partial}{\partial g_j}$, and the g_s expansion of F, namely

$$F = \sum_{g,c} g_s^{2g+c}F_{2g+c}\,, \tag{2.4}$$

to see that

$$\begin{aligned}\omega_0 &= \frac{d}{dV}(x)F_0 + \frac{S}{x}\\ \omega_1 &= \frac{d}{dV}(x)F_1\,.\end{aligned} \tag{2.5}$$

Then by demanding invariance of (2.1) under the transformation

$$\Phi \to \Phi + \epsilon \left(\frac{1}{(x-\Phi)} - \frac{1}{(x+\Phi)} \right), \tag{2.6}$$

we derive a loop equation for the full resolvent, the answer being

$$2 \oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x-x'} \omega(x') = \omega(x)^2 - \frac{g_s}{x}\omega(x) + g_s^2 \frac{d}{dV}(x)\omega(x) \,. \tag{2.7}$$

Using the g_s expansion of $\omega(x)$, we extract the zeroth and first order loop equations,

$$2 \oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x-x'} \omega_0(x') = \omega_0(x)^2 \,, \tag{2.8}$$

$$2 \oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x-x'} \omega_1(x') = 2\omega_1(x)\omega_0(x) - \frac{1}{x}\omega_0(x) \,. \tag{2.9}$$

Now one observes that a solution for $\omega_1(x)$ is given by,

$$\omega_1(x) = \frac{1}{2x} - \frac{1}{4}\frac{\partial \omega_0}{\partial S} \,. \tag{2.10}$$

The identity

$$\oint_{A_i} \frac{dx'}{2\pi i} \omega_1(x') = 0 \tag{2.11}$$

for each cycle A_i which encircles a cut then ensures that (2.10) is the unique solution to (2.9) (see [26] for a precise discussion on this point). Then using (2.5) we see that this implies

$$F_1 = -\frac{1}{4}\frac{\partial F_0}{\partial S} \,. \tag{2.12}$$

And so we have the first correction to the free energy in terms of the planar free energy, valid for any single trace potential.

2.2 A second look at F_1

The above method of obtaining F_1 does not generalize to the case of CS matrix models, as will become evident in section 5. So we will now find F_1 for this model using a rather different method than the one above, one which will generalize to the case of the CS matrix model.

We will assume there is just a single cut $(-a, a)$. We then modify the potential as follows,

$$\widehat{W}(x) = W(x) - \frac{g_s}{2}\ln(x) \,. \tag{2.13}$$

We will denote the free energy of this modified model $\widehat{F}$ and its resolvent $\widehat{\omega}(x)$. This modification effects the loop equation (2.7), by adding to the l.h.s. , a term

$$\frac{g_s}{2} \oint_C \frac{dx'}{2\pi i} \frac{\widehat{\omega}(x')}{x'(x-x')} \,, \tag{2.14}$$

where as usual, the contour does not encircle the point x. Now by deforming the contour to infinity we pick up the point x but the integral around infinity vanishes and we see that

$$\oint_{\mathcal{C}} \frac{dx'}{2\pi i} \frac{\widehat{\omega}(x')}{x'(x-x')} = \frac{\widehat{\omega}(x)}{x} \,. \tag{2.15}$$

So modifying the potential in this way leaves the leading order loop equation (2.8) unchanged, but the $\mathcal{O}(g_s)$ loop equation (2.9) becomes,

$$\oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x-x'} \widehat{\omega}_1(x') = \widehat{\omega}_1(x)\widehat{\omega}_0(x) \tag{2.16}$$

which has only the solution $\widehat{\omega}_1 = 0$. (any non-zero solution would also have scalar multiples as a solution and is thus unphysical). Another way to see it is notice that the loop equation (2.7) loses all terms proportional to g_s thus being reduced to one for $\mathrm{SU}(2M)$ which definitely has $\omega_1 = 0$. This implies that $\widehat{F}_1$ is a constant which can be taken to be 0.

The next step is to relate the free energy of the modified matrix model $\widehat{F}$ to the free energy of the unmodified model F. Since the leading order loop equation is unchanged, we have that $\widehat{F}_0 = F_0$. Now we will need to introduce the density function $\rho(x)$, it is related to the resolvent by

$$\omega(x) = \int_{-a}^{a} \frac{\rho_0(\lambda)}{x-\lambda} d\lambda \,. \tag{2.17}$$

Since we use the saddle point approximation, the subleading part of the free energy is equal to the subleading part of the matrix model action taken at the solutions of the equations of motion. We can see that

$$\begin{aligned} -\widehat{F}_1 &= \frac{1}{2}\int_{-a}^{a} \rho_0(\lambda) \ln\lambda d\lambda - F_1 & (2.18)\\ &= -\frac{1}{2}\int_{-a}^{a} \rho_0(\lambda) \left(\mathrm{P}\int_{0}^{\Lambda} \frac{1}{x-\lambda} dx \right) d\lambda - F_1 \,, & (2.19)\\ &= -\frac{1}{2}\int_{a}^{\Lambda} \omega_0(x) dx - F_1 \,, & (2.20)\\ &= -\frac{1}{4}\frac{\partial F_0}{\partial S} - F_1 \,. & (2.21) \end{aligned}$$

where Λ is some large cutoff and the principal value integral understood as $\mathrm{P}\int_0 = 1/2 \times (\int_{0+i\epsilon} + \int_{0-i\epsilon})$. Casting F_1 as an integral of a 1-form over a Riemann surface is natural in the context of matrix models with flavour which also has an $\mathcal{O}(g_s)$ correction to the free energy [22]. So (2.21) implies that

$$F_1 = -\frac{1}{4}\frac{\partial F_0}{\partial S} \,. \tag{2.22}$$

agreeing with the previous derivation of the same result. Generalizing this procedure to the CS matrix model will be the focus of this paper.

3. Topological string geometry

Chern-Simons theory on S^3 with the gauge group SU(M) describes topological A-branes wrapped around the S^3 in the deformed conifold T^*S^3. After the appropriate involution of the S^3 conifold geometry [18] the gauge group of the CS theory gets replaced by SO($2M$) or Sp($2M$) depending on the sign of the crosscap. The involution goes through the usual web of dualities and large N transitions. After the large N transition the closed string geometry is an orientifold of $O(-1)+O(-1)\to\mathbb{P}^1$, the $\mathbb{P}^1$ becoming an $\mathbb{RP}^2$. It is important that the involution does not have any fixed points. The mirror B-model geometry is again an orientifold of some deformed CY. The involution now has fixed points, two orientifold planes. This geometry has been also considered in [23]. It can be viewed as the end point of the large N transition on the B-model side when the B-model branes that are mirror to the A-model branes on S^3 disappear leaving only two orientifold planes. This two orientifold planes give a subleading contribution F_1 to the free energy that is not present for SU(M) gauge group. F_1 counts the holomorphic maps of $\mathbb{RP}^2$ into the resolved conifold.

After the canonical quantization the CS theory can be reduced to certain matrix model integrals [2]. Unlike usual hermitean matrix models where the integration is performed over the Lie algebra measure, the integration in the CS matrix model is over the Lie group. The matrix model also can be viewed as a result of canonical quantization of the HCS on the B-model side with a potential that contains double trace terms [3]. The mirror of T^*S^3 is given by the blownup of

$$xz=(e^u-1)(e^v-1)\,. \tag{3.1}$$

Clearly, the imaginary u direction is compact, with period $2\pi i$. The appearance of the group measure in the matrix integral can be interpreted as a result of the counting of all images of the D-branes in a matrix model with the Lie algebra measure [3]

$$\prod_n\prod_{i<j}(u_i-u_j+2\pi i n)^2\sim\prod_{i<j}\sinh^2\left(\frac{u_i-u_j}{2}\right). \tag{3.2}$$

If the gauge group is SO($2M$), each eigenvalue u_i of the matrix has its partner $-u_i$, so an additional product with the plus sign between eigenvalues appears in the measure.

It is possible to take the planar limit for this matrix model and obtain a spectral curve, in fact the spectral curve is the nontrivial part of the B-model geometry which is obtained after the large N transition. The leading contribution F_0 to the free energy is obtained from the integral of the resolvent over a noncompact (B) cycle. It is almost trivial to generalize this to SO($2M$) matrix model. The main concern of this paper will be the subleading part of the free energy (F_1),that is due the orientifold planes and is absent for SU(M) case. From the closed string point of view, the answer must be an integral of a meromorphic one-form from the orientifold planes to some fixed point at infinity. From matrix model point of view this one-form must be related to the resolvent.

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4. Free energy

In this section we review known results for the free energy and relate the parameters of topological string and the matrix model. The A-model orientifold of the conifold T^*S^3 has been considered before. In [18] the partition function of SO / Sp CS theory on S^3 was calculated to all orders in g_s and given a closed string interpretation. There it was found that $F_0^{\mathrm{SO}\,/\,\mathrm{Sp}} = 1/2F_0^{SU}$, i.e.

$$F_0^{\mathrm{SO}\,/\,\mathrm{Sp}}(t) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\exp^{-nt}}{n^3}\,. \tag{4.1}$$

The coefficient of $\frac{1}{2}$ is consistent with the orientifold action. From a closed string perspective, F_1 includes only the holomorphic maps of $\mathbb{RP}^2$ into the resolved conifold which are odd wrappings, i.e. only $\mathbb{Z}_2$ equivariant maps contribute to the instanton expansion. Furthermore the area of an $\mathbb{RP}^2$ instanton is half that of a S^2 instanton. So

$$F_1^{\mathrm{SO}\,/\,\mathrm{Sp}}(t) = \pm\sum_{n\ \mathrm{odd}}\frac{\exp^{-nt/2}}{n^2}\,, \tag{4.2}$$

where the $+(-)$ sign is for SO(Sp) respectively. In the SO($2M$) and Sp($2M$) matrix model, we will use a t 'Hooft parameter $S = g_sM$, related to the Kahler modulus t, the size of blown-up $\mathbb{RP}^2$, by

$$t = 2S \pm g_s\,. \tag{4.3}$$

By Taylor expanding $F_0^{CS}(2S \pm g_s)$ about $2S$, we see that F_0^{CS} (the SO / Sp CS free energy) contributes to F_1^{MM} (the matrix model with group measure free energy). We find the following identity,

$$F_0^{MM}(S) = F_0^{CS}(2S) \tag{4.4}$$

$$F_1^{MM}(S) = F_1^{CS}(2S) \mp \frac{1}{2}\frac{\partial F_0^{CS}(2S)}{\partial S}\,. \tag{4.5}$$

So more explicitly we have,

$$F_1^{MM}(S) = \pm\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-2nS}}{n^2} + \sum_{n\ \mathrm{odd}}\frac{e^{-nS}}{n^2}\right) \tag{4.6}$$

with $+(-)$ sign for SO(Sp) respectively. So we immediately see that the derivative relation (2.12) does not hold for the A-model, this was the observation which motivated the present work. The correct derivative relation is

$$F_1^{MM} = \pm\frac{1}{2}\left(\frac{\partial F_0^{MM}(S/2+i\pi)}{\partial S} + \frac{\partial F_0^{MM}(S/2)}{\partial S} + \frac{\partial F_0^{MM}(S)}{\partial S}\right) \tag{4.7}$$

and this will be verified in later sections. Note that the matrix model subleading free energy has two pieces (4.6). The first comes from the nontrivial relation between t 'Hooft parameter S and the Kahler modulus t, the second is the contribution of the orientifold planes.

Now another scenario where $\mathcal{O}(g_s^1)$ corrections appear is gauge theories with fundamental matter. In [22], the authors found that F_1 is an integral over the spectral curve of the *leading order* resolvent ω_0, consistent with what one would expect from [24, 25]. The fundamental matter shows itself in the matrix model as a subleading term in the potential. We will see that the contribution from the orientifold planes also comes in as a subleading term in the matrix model potential and therefore can also be expressed as an integral of the resolvent.

5. Loop equation

In this section we derive the loop equation needed to find the leading and subleading order resolvents, $\omega_0(z)$ and $\omega_1(z)$. Since the matrix integral is now over the Lie group rather than the Lie algebra the measure factor is different, and as a consequence the expression for the resolvent in terms of eigenvalues is different. We find that this does not greatly alter the derivative relation between $\omega_0(z)$ and $\omega_1(z)$ however we cannot use this to derive a relationship between F_0 and F_1 as we could for (2.12).

Since the matrices are in the adjoint of the gauge group, which for SO$(2N)$ is the antisymmetric representation, the partition function of the CS matrix model for SO$(2N)$ in eigenvalue form is

$$\mathcal{Z} \sim \int \prod_{i=1}^{M} du_i \prod_{j\neq i} \sinh^2\left(\frac{u_i - u_j}{2}\right) \sinh^2\left(\frac{u_i + u_j}{2}\right) \exp\left(-\frac{2}{g_s}\mathrm{W}(u_i)\right). \tag{5.1}$$

As for the matrix model with measure on the Lie algebra, the integral of the resolvent must be compatible with the log of the measure so we define,

$$\omega(x) \equiv g_s \left\langle \mathrm{Tr}\coth\left(\frac{x-\Phi}{2}\right)\right\rangle. \tag{5.2}$$

This can be written as,

$$\omega(x) = g_s \left\langle \sum_{i=1}^{M} \left(\coth(\frac{x-u_i}{2}) + \coth(\frac{x+u_i}{2})\right)\right\rangle \tag{5.3}$$

and it has asymptotic behaviour

$$\omega(x) = \pm 2S + \mathcal{O}(e^{\mp x}) \qquad x \to \pm\infty\,, \tag{5.4}$$

where we have defined $S = g_s M$. We restrict ourselves to one cut solutions, so we assume that $\omega(z)$ has one cut that runs from $-a$ to a along the real axis. With a potential given by $\mathrm{W}(x) = \sum_{j=1}^{\infty} \frac{g_j}{j} x^j$, the relationship between the resolvent and the free energy F is

$$\omega(x) = 2S\coth(\frac{x}{2}) + \frac{d}{dV(x)}F\,, \tag{5.5}$$

where the differential operator $d/dV(x)$ can be worked out by Taylor expanding $\coth(x-\Phi/2)$ around x and replacing Φ^j with $\partial/\partial g_j$. The resolvent and the free energy have expansions in g_s given by,

$$\begin{aligned} F &= \sum_{g,c} g_s^{2g+c} F_{2g+c}\,, \\ \omega(x) &= \sum_{g,c} g_s^{2g+c}\omega_{2g+c}(x)\,. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we see that,

$$\begin{aligned} \omega_0 &= \frac{d}{dV(x)}F_0 + 2S\coth\left(\frac{x}{2}\right), \\ \omega_j &= \frac{d}{dV(x)}F_j\,, \qquad j>0\,. \end{aligned} \tag{5.7}$$

The loop equation for this model can be derived by demanding reparametrisation invariance of the partition function (details are in the appendix). It is given by,

$$\frac{1}{2}\omega^2(x) - g_s\coth(x)\omega(x) + 2g_sS - 2S^2 - \widehat{\mathcal{K}}\omega(x) + \frac{g_s^2}{2}\frac{d}{dV(x)}\omega(x) = 0\,. \tag{5.8}$$

Where $\widehat{\mathcal{K}}$ acts as,

$$\widehat{\mathcal{K}}f(x) = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\coth\left(\frac{x-z}{2}\right)\mathrm{W}'(z)f(z)\,. \tag{5.9}$$

The contour $\mathcal{C}$ encircles the cut but not the point x. When we insert the expansion (5.6) into the loop equation (5.8), the first two equations we get are,

$$\mathcal{O}(g_s^0):\ \ \frac{1}{2}\omega_0^2(x) - 2S^2 = \widehat{\mathcal{K}}\omega_0(x)\,, \tag{5.10}$$

$$\mathcal{O}(g_s^1):\ \ \omega_1(x)\omega_0(x) + 2S - \coth(x)\omega_0(x) = \widehat{\mathcal{K}}\omega_1(x)\,. \tag{5.11}$$

From these equations we see that

$$\omega_1 = -\frac{1}{2}\frac{\partial\omega_0}{\partial S} + \coth(x) \tag{5.12}$$

is a solution to (5.11) and has the correct behavior at infinity ($\lim_{x\to\pm\infty}\omega_1(x)\sim\mathcal{O}(e^{\mp x})$). Similar equations for the Lie algebra case were found in [13, 14, 16, 17, 26]. This implies the following relation for the free energy,

$$\begin{aligned} \frac{d}{dV(x)}F_1 &= -\frac{1}{2}\frac{\partial\omega_0}{\partial S} + \coth(x) \\ &= -\frac{1}{2}\frac{d}{dV(x)}\frac{\partial F_0}{\partial S} - \coth(\frac{x}{2}) + \coth(x)\,. \end{aligned} \tag{5.13}$$

Here we see a significant difference between the matrix models with measure on the Lie algebra (see eq. (2.12)) or the Lie group. Here we cannot integrate to get F_1, doing so would amount to writing $\coth(x)-\coth(x/2)$ as $d/dV(x)$ of some function, something we were unable to do. Instead we will derive F_1 following the method we used in section 2.2 for the Lie algebra case. One needs to find a subleading term in the potential, the term that one has to subtract in order to kill the subleading part F_1 of the free energy. The first step to this goal is to find the leading order resolvent ω_0.

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6. Free energy of leading order

To find the resolvent $\omega_0(z)$ in the loop equation it is easier to go back and derive an equation of motion. Since we are mostly interested in the Chern-Simons matrix models we again restrict our attention to one cut solutions. Let introduce a density function $\rho_0(u)$ by,

$$\omega_0(z) = g_s \int_0^a \rho_0(u) \left(\coth \frac{z-u}{2} + \coth \frac{z+u}{2} \right) du \,. \tag{6.1}$$

It is convenient to continue the density function to the negative part of real axis $\rho(z)=\rho(-z)$. The above definition becomes

$$\omega_0(z) = g_s \int_{-a}^a \rho_0(u) \coth \frac{z-u}{2} du \,. \tag{6.2}$$

The normalization condition that guarantees the correct behavior of the resolvent at infinity is given by

$$\int_{-a}^a \rho_0(u) du = 2S \,. \tag{6.3}$$

Lets plug this definition into the loop equation for $\omega_0(z)$ (5.10). Subtracting the loop equation evaluated at the point $z + i\epsilon$ above the cut from the loop equation at the point $z - i\epsilon$ bellow the cut and taking into account that

$$\coth \frac{z-x-i\epsilon}{2} - \coth \frac{z-x+i\epsilon}{2} = 4\pi i \delta(z-x) \,, \tag{6.4}$$

one gets

$$\frac{2}{g_s} W'(z) = \mathrm{P} \int_{-a}^a \rho_0(u) \coth \frac{z-u}{2} du \,. \tag{6.5}$$

The usual way to proceed is to go to a new coordinate $U' = e^u$ [3]. In the case of SO$(2M)$ Chern-Simons matrix model, the potential is $W(z) = z^2/4$ and the equation of motion becomes

$$-\frac{1}{2g_s} \log(U e^{-2S}) = \mathrm{P} \int_{e^{-a}}^{e^a} \frac{\rho_0(U')}{U'-U} dU' \,, \tag{6.6}$$

where $U = e^z$. Here the normalization condition

$$g_s \int_{e^{-a}}^{e^a} \rho_0(U') \frac{dU'}{U'} = 2S \tag{6.7}$$

has been used. Following [3, 27] it is easy to find the function

$$v(U) = g_s \int_{e^{-a}}^{e^a} \frac{\rho_0(U')}{U'-U} dU' \,, \tag{6.8}$$

that satisfies the following

i) vanishes at infinity,

ii) has a square root cut,

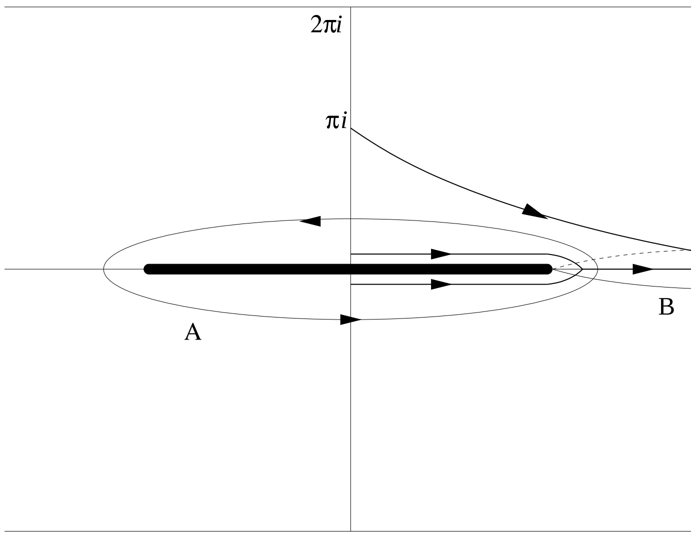

Figure 1: The physical sheet of the spectral curve.

iii) $v(0 - i\epsilon) = 2S$,

iv) $v(U - i\epsilon) + v(U + i\epsilon) = -\frac{1}{g_s}\log(Ue^{-2S})$.

The only difference from the SU(M) case considered in [3] is that S gets doubled,

$$v(U) = \log \frac{1 + U + \sqrt{(1+U)^2 - 4Ue^{2S}}}{2U}. \tag{6.9}$$

The relationship between this function and the resolvent is

$$\omega_0(z) = 2S - 2v(e^z)\,. \tag{6.10}$$

Let's discuss the geometry of the Riemann surface given by the resolvent $v(u)$. The spectral curve that corresponds to the resolvent is given by

$$e^v - e^{-u} + e^{-u-v+2S} - 1 = 0\,. \tag{6.11}$$

Since the resolvent has the property $v(u) = v(u + 2\pi i)$ the Riemann surface is compact in the imaginary direction. The resolvent has the square root cut giving rise to the two sheets of the surface. Therefore the Riemann surface looks like two infinite cylinders glued together along the cut. The contour around the cut is usually called an A cycle, the contour running from a point at infinity on one sheet to a point at infinity on the other sheet is called a B cycle. The Riemann surface is depicted in figure 1 and figure 2.

From the string theory point of view the curve is part of the mirror B-model geometry [23]. The branes have disappeared. In terms of type-IIA strings there are two orientifold O5-planes at points $u = \pm i\pi$. These will play a role in the next section when we calculate the $\mathcal{O}(g_s)$ part of the free energy. From the string theory point of view, the O5-plane con-

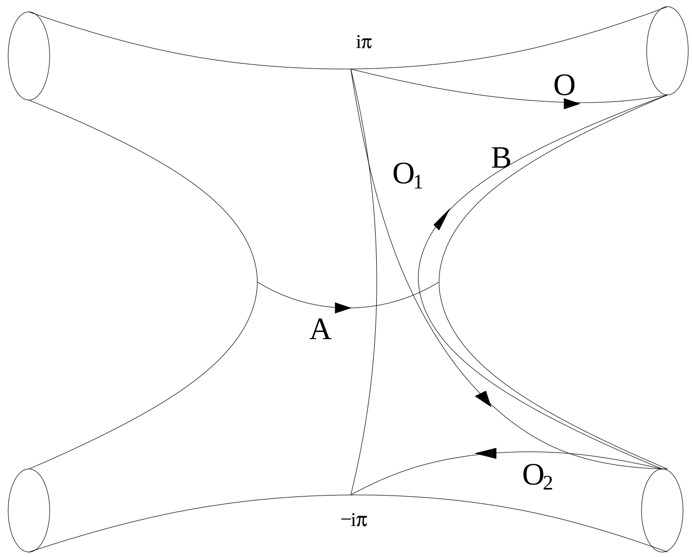

Figure 2: The Riemann surface and and the integration contours that give $\mathcal{O}(g_s)$ part of the free energy.

tribution to the superpotential is an integral from the location of the O5 plane to the point at infinity. The two O5-planes have different D-brane charges therefore their contributions are summed with opposite signs. We will see how these results emerge from the matrix model.

Now we are ready to calculate the leading contribution $F_0(S)$ to the free energy. As usual it is given by the integral of the resolvent over the B cycle. The integral can be expressed in terms of Euler's dilogarithm function (see appendix B for definition and useful properties)

$$\partial_S F_0(S) = \frac{1}{2}\int_B \omega_0(z)dz = -\int_B v(U)\frac{dU}{U} = -\mathrm{Li}_2\left(e^{-2S}\right). \tag{6.12}$$

Here all infinite and polynomial in S terms are omitted, leaving the worldsheet instanton contribution,

$$F_0(S) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-2nS}}{n^3}, \tag{6.13}$$

which agrees with [18]. The next objective is to calculate F_1 and cast it into the form of an integral over the resolvent.

7. Free energy of order $\mathcal{O}(g_s)$

The next objective is to calculate $\mathcal{O}(g_s)$ contribution to the free energy. Although there is a derivative relation between $\omega_0(z)$ and $\omega_1(z)$ (5.12), this cannot be integrated to a relation between F_0 and F_1 (5.13). Therefore to find the $\mathcal{O}(g_s)$ part of the free energy we use the new method that we have described in the second part of section 2.

7.1 F_1 as a dilogarithm

Let consider the $\mathcal{O}(g_s)$ term in the free energy. Since the saddle point method is used to construct the perturbative expansion in powers of g_s, there should not be any terms of order of g_s unless there is a subleading term in the effective matrix model action. To single out such a piece, we add the following subleading term to the action,

$$\frac{1}{g_s}\mathrm{Tr}\delta W(u) = \frac{1}{2}\sum_i \log\sinh^2 u_i\,. \tag{7.1}$$

This is analogous to (2.13) in the case of the Lie algebra matrix model. For this new potential, $W+\delta W$, we denote the free energy $\widehat{F}_1$. The equation for $\omega_0(z)$ (5.10) is invariant but the equation for $\omega_1(z)$ (5.11) becomes

$$\widehat{\mathcal{K}}(z)\widehat{\omega}_1 = \omega_0(z)\widehat{\omega}_1(z)\,. \tag{7.2}$$

As for the Lie algebra matrix models this integral equation has only the trivial solution $\widehat{\omega}_1 = 0$, which leads to $\widehat{F}_1 = 0$. This suggests that (7.1) cancels the subleading part of the action. It is also can be seen from the fact that the terms proportional to g_s in the full loop equation (5.8) disappear leaving us with the loop equation for $\mathrm{SU}(2M)$ (A.10) which has $F_1 = 0$.

So, similar to (2.21), we have,

$$0 = -\widehat{F}_1 = -F_1 + \frac{1}{2}\int_0^a \rho_0(z)\log\sinh^2 z dz\,. \tag{7.3}$$

In principle, if one knows the density function $\rho_0(z)$, the integral can be taken. It is more convenient however to have it written as an integral of $\omega_0(z)$, which is known. We will see that written in that form, F_1 has two different pieces, corresponding to the integration along different contours.

It is easy to see that

$$\log\sinh z = \log 2 + \log\sinh\frac{z}{2} + \log\sinh\frac{z+i\pi}{2} - \frac{i\pi}{2}\,. \tag{7.4}$$

In what follows we omit all infinite terms and polynomial terms that can be easily restored. Similar to [22], we write the logarithms in the integral (7.3) as

$$\log\sinh\frac{z}{2} = -\frac{1}{2}\mathrm{P}\int_0^\Lambda \coth\frac{x-z}{2}dx - \frac{z}{2} \tag{7.5}$$

$$\log\sinh\frac{z+i\pi}{2} = -\frac{1}{2}\int_{i\pi}^\Lambda \coth\frac{x+z}{2}dx + \frac{z}{2} \tag{7.6}$$

where Λ is a point at infinity, a UV cutoff. Now one can recognize the resolvent in (7.3). Combining the above and using the fact that $\rho_0(z)$ is an even function we get

$$F_1(S) = -\frac{1}{4}\left\{\mathrm{P}\int_0^\Lambda \omega_0(z)dz + \int_{i\pi}^\Lambda \omega_0(z)dz\right\}. \tag{7.7}$$

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To figure out which branch of the function $\omega_0(z)$ is to be used in the above integrals or in other words on which sheet the point Λ is located, one has to look at the behavior at infinity of the integrals of the $\coth(z/2)$ function. The conclusion is that Λ is a point on the physical sheet where $\omega_0(z)$ vanishes at infinity. The principal value integral is to be understood as $\mathrm{P}\int_0 = 1/2(\int_{0+i\epsilon} + \int_{0-i\epsilon})$. The two contours are drawn on figure 1. So far our discussion in this section has been applicable to one cut solutions of the matrix model with arbitrary potentials. From this point we restrict ourselves to quadratic potentials only.

The first integral in (7.7) can be written as an integral over the B cycle. To see this, we introduce a function $y(z)$ that corresponds to the singular part of the resolvent, so the integrals over a cycle of $y(z)$ or of the resolvent are the same. The function $y(z)$ has the property of a square root, i.e. it changes the sign when passing over the cut, $y(z+i\epsilon) = -y(z-i\epsilon)$. The differential $y(z)dz$ is the meromorphic one-form on the Riemann surface defined by the resolvent. One has

$$\omega_0(z) = -2W'(z) + y(z)\,, \tag{7.8}$$

$$-\frac{1}{4}\mathrm{P}\int_0^\Lambda \omega_0(z)dz = -\frac{1}{8}\int_B \omega_0(z)dz + \frac{1}{2}W(\Lambda)\,. \tag{7.9}$$

To check the result (7.7) we calculate $F_1(S)$ for the quadratic potential and compare it with the known result. The integral over the B cycle has been taken already in the previous section. Therefore

$$-\frac{1}{4}\mathrm{P}\int_0^\Lambda \omega_0(z)dz = \frac{1}{4}\mathrm{Li}_2\left(e^{-2S}\right). \tag{7.10}$$

The second integral in (7.7) can also be expressed in terms of Euler's dilogarithm function (see appendix B for the details),

$$-\frac{1}{4}\int_{i\pi}^\Lambda \omega_0(z)dz = \frac{1}{4}\mathrm{Li}_2(e^{-2S}) + \frac{1}{2}\left(\mathrm{Li}_2(e^{-S}) - \mathrm{Li}_2(-e^{-S})\right). \tag{7.11}$$

where again we dropped all polynomial terms. After summing (7.10) and (7.11) one gets the correct free energy $F_1(S)$ (4.6).

7.2 F_1 as a contour integral

There is another way of writing (7.11). In [23], the contribution of O5-planes was found in the form of integrals over the spectral curve from positions of the O5-planes to some points at infinities. We want to separate clearly the part coming from O5-planes and the part from the expansion of $F_0(t)$ (see the comment after eq (4.7)). The last one is just the integral over the B cycle. Again it is convenient to appeal to the meromorphic one-form $y(z)dz$

$$\int_{i\pi}^\Lambda \omega_0(z)dz = \frac{1}{2}\left(\int_{i\pi}^\Lambda y(z)dz + \int_{\Lambda'}^{-i\pi} y(z)dz\right). \tag{7.12}$$

The contour in the second integral is on the second sheet of the Riemann surface, and Λ' is a point at infinity on this sheet (see figure 2). Let denote the contour in the first integral as O and in the second as O_2. From figure 2 it is clear that $O = O_1 + B$ where the contour O_1 goes from $i\pi$ to Λ'. The conclusion is

$$\int_{i\pi}^{\Lambda} \omega_0(z)dz = \frac{1}{2}\left(\int_B \omega_0(z)dz + \int_{O_1} \omega_0(z)dz + \int_{O_2} \omega_0(z)dz\right). \tag{7.13}$$

The contour O_1 and the contour O_2 run from the positions of two O5-planes to the point at infinity which is precisely what is expected from the string theory point of view. This consideration so far is valid for arbitrary potentials. If the potential is quadratic one has to take integrals of $v(u)$. The change of variables $v(u)du = vu'(v)dv$ brings the integrals over O_1 and O_2 contours to the same form as in [23].

Almost the same calculation can be done for the case of Sp$(2M)$ group. The group measure for Sp$(2M)$ has the extra factor $\prod_i \sinh^2 u_i$ which corresponds to the additional term $\sum_i \log\sinh^2 u_i$ in the matrix model effective action. To have $\widehat{F}_1 = 0$ the term

$$\frac{1}{g_s}\mathrm{Tr}\delta\mathrm{W}(u) = -\frac{1}{2}\sum_i \log\sinh^2 u_i \tag{7.14}$$

has to be added to the potential. It is clear now that the only difference from SO$(2M)$ case is the opposite sign of $F_1(S)$ which agrees with (4.6).

While it is possible to continue the Chern-Simons partition function to SO$(2M+1)$ [18] and take the large M limit, this partition function does not match the partition function of the SO$(2M+1)$ matrix model. So although the SO$(2M+1)$ matrix model does not corresponds to Chern-Simons theory one can still consider this model. The group measure has an extra factor $\prod_i \sinh^2 \frac{u_i}{2}$. Repeating the above procedure one gets

$$F_1 = \frac{1}{2}\sum_i\left(\log 2 - \frac{i\pi}{2} + \log\sinh^2\frac{u_i + i\pi}{2} - \log\sinh^2\frac{u_i}{2}\right). \tag{7.15}$$

Since the second logarithmic function has changed sign compared to the SO$(2M)$ case the two integrals over the B cycle cancel each other and we are left with integrals over the contours O_1 and O_2. So the part that is proportional to the derivative of F_0 disappears. This is in contrast with the Lie algebra case, in which the SO$(2M+1)$ and Sp$(2M)$ matrix models have the same free energy.

8. $\mathcal{N} = 1$ SYM in four dimensions

The tree level superpotential is figured out (following [2]) by converting the Haar measure into a measure on the Lie Algebra

$$\prod_{i<j}\left(2\sinh\frac{u_i - u_j}{2}\right)^2\left(2\sinh\frac{u_i + u_j}{2}\right)^2 = \prod_{i<j}(u_i^2 - u_j^2)^2 \exp\left(\sum_{k=1}^{\infty} a_k\left(\sigma_k^- + \sigma_k^+\right)(u)\right), \tag{8.1}$$

where,

$$\sigma_k^{\pm}(u) = \sum_{i<j}(u_i \pm u_j)^{2k}, \qquad a_k = \frac{B_{2k}}{k(2k)!}, \tag{8.2}$$

B_{2k} are the Bernoulli numbers. Using the fact that the Newton polynomials

$$P_k(u) = \sum_{i=1}^{M} u_i^k\,, \tag{8.3}$$

are equal to $\frac{1}{2}\mathrm{Tr}M^k$, where M is an anti-symmetric matrix gauge fixed to the diagonal, we see that the four dimensional tree-level superpotential engineered by this construction is

$$W_{\mathrm{tree}}(\Phi) = \frac{1}{8}\Phi^2 - \frac{g_s}{2}\sum_{k=0}^{\infty} a_k \left[-2^{2k-2}\Phi^{2k} + \sum_{s=0}^{2k} \left(2k s \right) \frac{1}{4N}\mathrm{Tr}\Phi^s\mathrm{Tr}\Phi^{2k-s}(1+(-1)^s)\right]. \tag{8.4}$$

Only even powers of Φ appear, as expected.

The Chern-Simon's partition function will give the four dimensional low energy Wilsonian effective action for $\mathcal{N}=1$ SYM with W_{tree} given by (8.4). Now in [18], the correct closed string variable was identified as $t = g_s(2M-1)$ (for SO$(2M)$), which is identified with the gluino condensate. So following [18, 13] we propose the formula

$$W_{\mathrm{eff}} = \mathcal{Q}_{D6}\frac{\partial F_0^{CS}}{\partial t} + \mathcal{Q}_{O6}G_0^{CS} - \tau t\,, \tag{8.5}$$

where $\mathcal{Q}_{D6}$ is the total D6-brane charge and $\mathcal{Q}_{O6}$ is the total O6-plane charge, τ is the gauge coupling. Op-plane charge is given by $\pm 2^{5-p}$, -sign for SO, +sign for Sp. We have introduced $G_0 = aF_1^{CS}$, a is a constant. It is important to use F^{CS} not F^{MM} in 8.5 because their g_s expansions differ, as discussed in the introduction.

To make contact with results of [28], we look at log(vol(SO$(2M)$). This term is already within the free energy but we know from [29] that it is this term which supplies the $t\log t$ term to the superpotential. So, expanding in $(2M-1)$ and keeping only log terms, we get

$$-\log(\mathrm{vol}(\mathrm{SO}(2M)) \sim g_s^{-2}\frac{t^2}{4}\log t + g_s^{-1}\frac{t}{4}\log t + \cdots \tag{8.6}$$

so using (8.5), we find

$$W_{\mathrm{eff}} = \frac{N}{4}t\log t - \frac{1}{2}\frac{at}{4}\log t + \frac{N}{2}\frac{\partial F_0^{pert}}{\partial t} - 2F_1^{pert} - \tau t\,, \tag{8.7}$$

where $F^{pert} = F + \log(\mathrm{vol}(\mathrm{SO}(2M))$. So requiring $N-2$ vacua, we find that $a=4$.

9. Conclusion

We have studied the matrix models with Haar measure on SO / Sp in the large M limit, for which we have introduced the new form of the resolvent that is compatible with the group measure. We have derived a quantum loop equation and for the case of quadratic potential, and have found the leading order resolvent.

We have calculated the $\mathcal{O}(g_s)$ corrections to the SO and Sp gaussian matrix model free energy using a novel method. This method separates in a clear way the leading (of order $\mathcal{O}(g_s^{-1})$) and subleading (of order $\mathcal{O}(g_s^0)$) parts in the effective matrix model action. The free energy of the first two orders was expressed as integrals of the leading order resolvent over the spectral curve and these integrals were explicitly performed.

We have found agreement between matrix model and large M Chern-Simons results. While the g_s expansion of the Chern-Simons theory has a nice worldsheet interpretation, which means that the first two orders correspond to sphere and $\mathbb{RP}^2$ worldsheets, the g_s expansion of the matrix model free energy mixes the worldsheet contribution at each order, essentially due to a shift in the identification of the 't Hooft parameter. The derivative relation found in Lie algebra matrix models

$$F_1 = -\frac{1}{4}\frac{\partial F_0(S)}{\partial S} \tag{9.1}$$

does not hold but instead we find that

$$F_1 = -\frac{1}{2}\left(\frac{\partial F_0(S/2+i\pi)}{\partial S} + \frac{\partial F_0(S/2)}{\partial S} + \frac{\partial F_0(S)}{\partial S}\right). \tag{9.2}$$

Here F_1 contains a contribution from sphere worldsheets as well as $\mathbb{RP}^2$ worldsheets.

Type-IIA string theory on $M^{3,1} \times T^*S^3$ with the internal geometry orientifolded and N D6 branes wrapped on $M^{3,1} \times S^3$ engineers an $\mathcal{N} = 1$ SO / Sp SYM in four dimensions with a certain double trace tree level superpotential which was given. The calculation of the leading and subleading free energy in these matrix models or equivalently in Chern-Simons theory gives the effective superpotential for this four dimensional SYM. This was also discussed.

Although we have presented the main results with the potential $W(z)$ an arbitrary polynomial, the string theory application of these matrix models is known only for the quadratic potential. One can convert the matrix model to a Lie algebra matrix model with double trace potential and a single trace potential $W(\Phi)$. This potential then corresponds to the tree level superpotential of an $\mathcal{N} = 1$ SYM in four dimensions. Therefore one knows the four dimensional effective theory but does know the internal geometry which constructs this theory. If the potential has higher than quadratic powers, the spectral curve will not be a polynomial in e^u and e^v. Whether or not this spectral curve can be related to some B-model geometry is an interesting question to address.

It would be interesting to generalize the impressive work [30] and solve the SO / Sp matrix model to all orders by the method of orthogonal polynomials.

Acknowledgments

We are particularly grateful to Mina Aganagic for bringing [22] to our attention. We also thank Marcos Marino for explaining his work to us. Thanks are due to Nick Warner for the reading the manuscript. We have much enjoyed the interaction with all members of the string group at USC. NH is supported by a Fletcher Jones Graduate Fellowship.

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A. Derivation Of the Loop Equation

In this paper we are just interested in calculating the free energy contribution from $\mathbb{RP}^2$ worldsheets. To do this we derive an all worldsheet loop equation. This equation will also be derived for Lie group SU(M) for completeness.

A.1 SO($2M$)

The partition function, after integrating out the off diagonal terms, is given by,

$$\mathcal{Z} \sim \int \prod_{i=1}^{M} du_i \prod_{j \neq i} \sinh^2\left(\frac{u_i - u_j}{2}\right) \sinh^2\left(\frac{u_i + u_j}{2}\right) \exp\left(-\frac{2}{g_s}\mathrm{W}(u_i)\right). \tag{A.1}$$

We perform the infinitessimal change of co-ordintes,

$$u_i \to u_i + \epsilon\left(\coth\left(\frac{x - u_i}{2}\right) - \coth\left(\frac{x + u_i}{2}\right)\right). \tag{A.2}$$

and demand that the partition function is invariant under this transformation. At first order this yields the following constraint,

$$\begin{aligned} \sum_{i=1}^{M} \Bigg\langle \frac{1}{2}\left(\mathrm{cosech}^2\left(\frac{x - u_i}{2}\right) + \mathrm{cosech}^2\left(\frac{x + u_i}{2}\right)\right) + \\ + \sum_{j \neq i}\left(\coth\left(\frac{x - u_i}{2}\right) - \coth\left(\frac{x + u_i}{2}\right)\right)\left(\coth\left(\frac{u_i - u_j}{2}\right) + \coth\left(\frac{u_i + u_j}{2}\right)\right) - \\ - \frac{2}{g_s}\left(\coth\left(\frac{x - u_i}{2}\right) - \coth\left(\frac{x + u_i}{2}\right)\right)\mathrm{W}'(u_i)\Bigg\rangle = 0. \end{aligned} \tag{A.3}$$

Now we need the 2 identities,

$$\begin{aligned} \sum_{i,j=1}^{M} \frac{\cosh(\frac{u_i - u_j}{2})}{\sinh(\frac{x - u_i}{2})\sinh(\frac{x - u_j}{2})} &= \sum_{i=1}^{M} \mathrm{cosech}\left(\frac{x - u_i}{2}\right)\mathrm{cosech}\left(\frac{x - u_j}{2}\right) + \\ &\quad + 2\sum_{i \neq j} \coth\left(\frac{x - u_i}{2}\right)\coth\left(\frac{u_i - u_j}{2}\right) \\ &= \sum_{i,j=1}^{M} \coth\left(\frac{x - u_i}{2}\right)\coth\left(\frac{x - u_j}{2}\right) - M^2. \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} \sum_{i,j=1}^{M} \frac{\cosh(\frac{u_i - u_j}{2})}{\sinh(\frac{x + u_i}{2})\sinh(\frac{x + u_j}{2})} &= \sum_{i=1}^{M} \mathrm{cosech}\left(\frac{x + u_i}{2}\right)\mathrm{cosech}\left(\frac{x + u_j}{2}\right) - \\ &\quad - 2\sum_{i \neq j} \coth\left(\frac{x + u_i}{2}\right)\coth\left(\frac{u_i - u_j}{2}\right) \\ &= \sum_{i,j=1}^{M} \coth\left(\frac{x + u_i}{2}\right)\coth\left(\frac{x + u_j}{2}\right) - M^2. \end{aligned} \tag{A.5}$$

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With a little more work, one can also show that

$$\frac{1}{2}\omega^2(x) - g_s \coth(x)\omega(x) + g_s S =$$
$$= \frac{g_s^2}{2}\sum_{i,j=1}^{M}\left[\coth\left(\frac{x+u_i}{2}\right)\coth\left(\frac{x+u_j}{2}\right) + \coth\left(\frac{x-u_i}{2}\right)\coth\left(\frac{x-u_j}{2}\right)\right] +$$
$$+ g_s^2 \sum_{i\neq j}\left(\coth\left(\frac{x-u_i}{2}\right) - \coth\left(\frac{x+u_j}{2}\right)\right)\coth\left(\frac{u_i+u_j}{2}\right) + S^2 - g_s M \,. \quad \text{(A.6)}$$

Now we multiply (A.3) through by g_s^2 and employ (A.4), (A.5), (A.6) to get the final form of the loop equation,

$$\frac{1}{2}\omega^2(x) - g_s \coth(x)\omega(x) + 2g_s S - 2S^2 - \widehat{\mathcal{K}}\omega(x) + \frac{g_s^2}{2}\frac{d}{dV(x)}\omega(x) = 0\,, \quad \text{(A.7)}$$

where the linear operator $\widehat{\mathcal{K}}$ acts as

$$\widehat{\mathcal{K}} f(x) = \oint_{\mathcal{C}} \frac{dz}{2\pi i}\coth\left(\frac{x-z}{2}\right) \mathrm{W}'(z) f(z)\,. \quad \text{(A.8)}$$

A.2 SU(M)

Just as above, we integrate out the off diagonal components and demand reparametrisation invariance under

$$u_i \to u_i + \epsilon \coth\left(\frac{x-u_i}{2}\right). \quad \text{(A.9)}$$

Since the derivation is much simpler than for SO($2M$), we just state the result,

$$\frac{1}{2}\omega^2(x) - 2S^2 - \frac{1}{2}\widehat{\mathcal{K}}\omega(x) + \frac{g_s^2}{2}\frac{d}{dV(x)}\omega(x) = 0\,. \quad \text{(A.10)}$$

B. Dilogarithm identities

Euler's dilogarithm function Li_2 is defined as the integral

$$\mathrm{Li}_2(z) = \int_z^0 \frac{\log(1-t)}{t} dt\,, \quad \text{(B.1)}$$

or as the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^k}{k^2}\,. \quad \text{(B.2)}$$

Among the many amazing properties of this function, we will use the following (for a review see for example [31])

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2)\,, \quad \text{(B.3)}$$
$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2(1-z^{-1}) = \frac{1}{2}\log^2 z\,, \quad \text{(B.4)}$$
$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - (\log z)(\log(1-z))\,. \quad \text{(B.5)}$$

The integral from $i\pi$ to Λ can be taken, with the result

$$\frac{1}{4}\int_{i\pi}^{\Lambda}\omega_0(z)dz = -\frac{1}{2}\bigg(\frac{1}{2}\log^2\left(-e^{-S}\right) - \log\left(-e^{S}\right)\log\left(1+e^{-S}\right) + \\ + \mathrm{Li}_2\left(e^{-2S}\right) - \mathrm{Li}_2\left(-e^{-S}\right) - \mathrm{Li}_2\left(1+e^{S}\right)\bigg)\,. \qquad \text{(B.6)}$$

To cancel the product of logarithms one has to apply (B.4) to $\mathrm{Li}_2(1+e^S)$ to get $\mathrm{Li}_2(1+e^{-S})$, then using (B.5) convert it to $\mathrm{Li}_2\left(-e^{-S}\right)$. After this manipulation and omitting all polynomial terms in S, one has

$$\frac{1}{4}\int_{i\pi}^{\Lambda}\omega_0(z)dz = -\frac{1}{2}\left(-2\mathrm{Li}_2\left(-e^{-S}\right) + \mathrm{Li}_2\left(e^{-2S}\right)\right). \qquad \text{(B.7)}$$

The last step is to apply the property (B.3) and get (7.11).

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